

A NEW MATHEMATICAL MODEL TO DETERMINE LOSS IN INTERCONNECTED INDUSTRY SYSTEMS

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ABSTRACT

In this paper, the terms disaster is conceptualised and explained as an event or a group of events with the corresponding outcomes. An event is described in detail by its severity and size. The corresponding numerical outcome of the event is redefined by magnitude of damage. The damage may be categorised as physical, social, and psychological and the consequential aspect of damage may be shown as the economic loss. The loss can be the output degradation of one industry of the economy. The relationships between the various aspects of the damage mentioned are clarified in this paper. The output-degradation may spread to other industries in an economy through the interaction linkages between the affected industry and other industries. In this paper, a new mathematical model is developed based on the input-output model. The model can measure the economic loss via interconnected linkage and the parameters of the model are estimated by three innovative methods. The stability, limitation, and application of the model are also discussed and the event size distribution is considered as an additional random factor. Therefore a random distribution is linked to the model and the model is tested with a numerical example.

Background

The increasing frequency and severity of natural and human-caused disasters affects stability, growth, and prosperity of an economy as well as the well being of society. These disasters are more significant when they are unexpected and uncertain. Regardless of whether this uncertainty has a random nature or is associated with inaccuracy in predictions, the consequences are costly and unavoidable. Minimizing the loss and costs of events or disasters is an ambition of both public and private decision makers and planners. The aim of this paper is to develop a new model that improves upon the existing input-output model to determine economic loss in interconnected industrial systems.

Quantifying uncertainty to avoid or reduce the costs of such events can be effectively achieved through an engineering impact analysis. Engineering impact analysis enables different groups involved in key decision making assess probable and/or possible occurrence of disasters and therefore facilitate appropriate precautions. This assessment might lead to improved stability of the economy and social indices in a proposed region through investment and allocation of appropriate resources. Specifically examining the probable economic impacts of uncertain natural and human-caused disasters, it is important to recognise all elements and components of an economy and their interconnections, the major sources of uncertainty, the probable and/or possible impacts on the elements and components, and spreading of disasters to intact components through physical and functional interdependencies.

Developing a logical framework with the aim to model and formulate the economic impact of disaster relies on the concept of the term *event*. The term event is defined as a phenomena associated with numerical outcomes. These numerical outcomes may be measured to be the probability and possibly of the corresponding damages and related physical outputs. The numerical outcomes may be also classified by the occurrence of events, magnitude of events, and etc. Therefore, probability theory is one of the fundamental principles of such a logical framework. In addition, in an economy with interconnected elements and components, any possible external or internal event on one element may impact all other elements and subsequently the economy. An economy acts as a system and is the union of elements and components including all possible interaction links. In particular, the interactions links causes the spreading and cascading of the event consequences. The term *element* in an economy refers to a particular branch of economy or commercial activity. Based on this definition, the term *element* is substituted by the term *industry* in this paper.

Disaster, event and corresponding outcomes

Several researchers have conceptualised the term *disaster* (Barkun, 1974; Dynes, 1976; Quarantelli, 1985). Barkun (1974) classified disaster as *physical, social, and psychological* damages. Dynes (1976) suggested disaster as the potential aspect of *something* that can generate physical impacts on the environment. The physical impacts are classified as *natural* or *human caused* agents in most disaster related research. Quarantelli (1985) classified and equated disaster using seven ideal-type terms but out of these only three are relevant to this study; namely, physical agents, physical impact of such agents, and assessment of physical impacts.

Disaster is also defined as *an imbalance in the demand-capability ratio in a crisis occasion* (Quarantelli 1985, p. 50). Pelling (2003) described disaster as the disruption to how a system functions; while the system may operate from small scale (such as individuals' biological and psychological constitutions) to large scale (e.g. urban infrastructure network). In all attempts to define the term *disaster* the critical aspect is that measure of a disaster's unwelcome consequential impacts. For this purpose, this term and it consequential

impacts have been also described by the term *event*. Therefore, the term *event* refers to both concepts of disaster and related quantitative outcomes. The first conceivable quantitative outcome can be the event's occurrence frequency. For example, changing climate and increasing the risk associated with terrorism, both may logically raise the frequency of events. It needs to be also noted that the occurrence of events may not be normally distributed and so sometimes it needs to be studied through a geological or geopolitical approach.

Newly generated policies have emanated from studies on regulation and standardisation of constructions, network, immigration, or allocation. Some studies have been done on increasing the frequency of events and their uncertainty (Masika 2002, Wisner *et al.* 2004, Clark *et al.* 2006). For example, Wisner *et al.* (2004) studied human populations as a function of an event's frequency, while Masika (2002) noted the weakening of recovery over livelihood systems after facing uncertain and frequent events. Additionally, the number of events is an important factor for event assessment. It is possible that more than one event occurs at the same time, or an event generates others. For example, the impact of an earthquake and heatwave at the same time increases the stress on vulnerable people (Earth policy institute, 2003); Scawthorn (2000) remarked on the weakness in adequate fire fighting capability caused by an earthquake during a drought. Managing the combinations of events may be achieved by developing a logical interaction with the probability related to the events. In this regard, Ang and Tang (2007) quantified the relationship between events by rules such as addition, multiplication, total probability, or Bayes' theorem.

The second quantitatively measurable factor concerns the severity of events and consequential damage. This measurement can be related to the frequency of severity or magnitude of damage but combinations of events also need to be considered. Basoz and Kiremidjian (1998) constructed empirical fragility curves based on the reported and aggregated data from past earthquake damage. Several fragility curves have also been developed such as fragility curves for bridges (Hwang *et al.*, 2001), and power and water systems (Shinozuka *et al.*, 2004). While, fragility curves can be verified by the availability of historical and current data, in the absence of such data for events the only feasible approach is to develop these curves theoretically. In fact, fragility curves are a conditional type of probability. These curves provide the likelihood of a certain level of damages that a structure will experience or exceed (Nielson, 2003). The data can be aggregated based on natural randomness or artificially generated by proposed boundaries to reify theoretical models.

Therefore, the term event may be defined as a disaster with some significant measurable outcomes such as size and the rate of occurrence. A proper data set of the outcomes may support the relevant groups of people to estimate the consequences of probable and/or possible events including corresponding damage.

Event, severity and damages

One area of engineering research assesses events by their level of extremity, such as maximum earthquake intensity. The term extreme refers to both the maximum and minimum of a set of numerical outcomes of an event. As long as measurement term is numerical, it can be studied in this manner. For example in statistics, quantifying extremity is related to the largest and the smallest values (extreme) in a sample, and the sample size. The outcome distribution of consequences therefore may be assumed to be an exact or follow an asymptotic distribution (Ang and Tang, 2007). The extreme values in an uncertain extreme event (such as an extreme natural event) may not be predictable before occurrence of that event. Also, the values may not be achievable from historical data. Hence, to evaluate the extremity in pre-event situation, a set of random distributions related to occurrence of extreme events can be constructed using a set of assumptions for the purpose of analysis.

The corresponding damage of an extreme event may be shown by a distribution with lower and upper boundaries. In such a distribution, the population of damage varies between 0 and 1 (0 and 100% destruction). It may be illustrated by *Cumulative Distribution Function* (CDF) curves in terms of initial variates (Figure 1). The sample size (n) can also affect the distribution. For example, when $n = 1$ the distribution becomes an exponential curve. However, larger sample size (n) increases the fidelity of the estimated distribution to the *real* one and changes the curve to an *S-shape*. This means that the growth pattern can be defined by an S-shaped curve Initially, the population increases slowly with a positive acceleration; then increases rather rapidly to approach exponential growth rate: labelled the *J-shape*; then when the growth in population declines because of a negative acceleration; but the growth rate is nevertheless still positive; and finally the population gains its maximum value (100%) with a zero growth rate. In the absence of historical data, the two parameters: namely, sample size (n) and population of damages can be estimated through the conduction of a set of *what-If scenarios*. The larger sample size (n) may indeed result in better accuracy; for example, Figure 1 shows that when the sample size n is larger than 1, the distribution moves from an exponential to an *S-shape* (As noted earlier, the *S-shape* appears more realistic when compared to others in terms of damage and extremity).

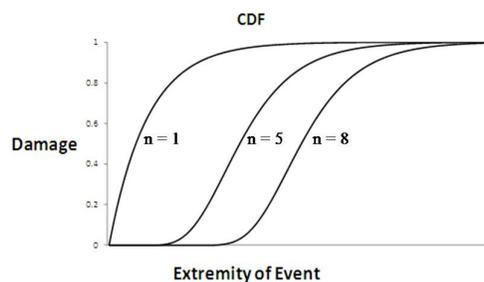


Figure 1: Distribution of damages versus extremity of event for different values of n

Before assessing damage, the intersection among events needs to be also considered. The simultaneous occurrence of more than one extreme event, or the occurrence of a series of extreme events, may propagate damages. For example, explosions in gas pipelines or floods due to failed dam may occur after an earthquake. In this case, earthquake as an event generates or causes extra set of events that follow. Figure 2 shows possible interactions between two extreme events and the resultant overall damages flow chart.

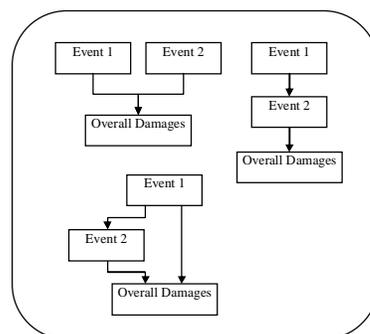


Figure 2: The nature of overall damages in case of two extreme events

The overall damage may also be modelled based on the distribution of damage (Figure 1). In this regard, all interactions between events need to be evaluated through a large database (Figure 3). While in this research the quantification of damage is predicted through a set of *what-If* scenarios, the effect of events are provided as a package of so-called *overall damages*. It is critical to note that in aggregating the impacts of damage, the maximum damage cannot exceed 1 (100%).

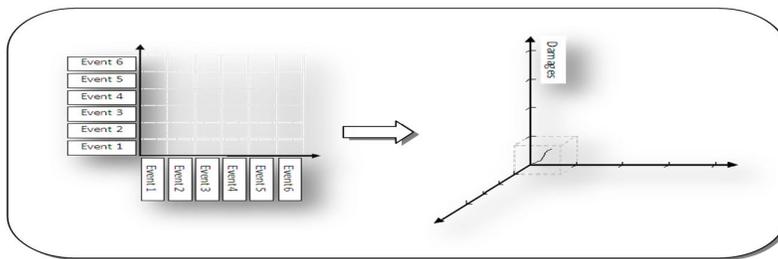


Figure 3: Conceptual model of overall damages of six simultaneous extreme events

Accepting the classifications of damage (*physical impact*, *social impact*, and *psychological impact*) provided by Barkun (1974) (Figure 4), the assessment of the loss, associated with an event needs to be determined for each type. It is a fact that in most cases, the three types intersect but initially to avoid complexity in modelling it may be assumed that there is no interaction between the impacts (the impact of any two of them is zero). Essentially, then if the impact of social and psychological is assumed to be zero, the physical impact adequately captures the estimated total loss.

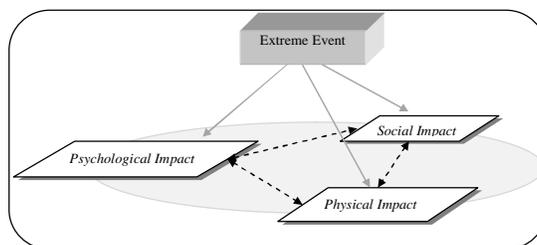


Figure 4: Classification the impact of extreme events

Physical impact

An important consideration when developing a disaster framework is the economic loss. An economy is a system that consists of a number of physically interconnected industries. Each industry acts as a provider of commodities to others (*supplier*) and/or consumers (*end-users*). Extreme events may potentially impact the industries and/or their linkages physically. This physical impact for example, may be a destroyed powerhouse or a set of broken power-lines. The obvious consequence of physical impact on an industry is *functional disorder*. This functional disorder may simultaneously impact upon the industry outputs.

The physical impact, based on the above explanation, can be defined as degradation of the industry outputs. This degradation may be measured by the percentage of the output (*over as-planned* output) and/or outage of the industries. In both cases, the results are the same: perturbation in the supply and demand chain. The actual physical impact measurement is rather difficult to obtain. The process needs a large database for every possible event and its effect on each industry. However, when the goal is to measure the economic loss, the physical impact may be determined by aggregating the output degradation of affected industries as

well as their cascading effects on interconnected industries. A study on the monetary flow between the industries may be most efficient way to calculate the physical impact.

Social impact

Social impact may simply be defined by the impact of events on society but a better definition may be found through the assessment of this impact. Burdge (1995) defined the social impact assessment as an approach to identify and assess the impact of change on society including individuals, families, and communities. Becker (2001) described social impact assessment by the act of understanding and recognising the future consequences of an action related to individuals, organizations and social macro-systems. Social impact is related to the future consequences of any social-related actions. Chadwick (2002) indicated that most of social impact assessments are focused on quantitative factors such as population changes, infrastructure needs, and economic assessments. Therefore, it can be concluded that social impact is an existing and/or potential (future consequences) impacts on society and it may be measured by numerical outcomes.

Haas et al. (1977) defined four steps that a community must undergo after a disastrous event: (i) the emergency period; (ii) restoration period; (iii) replacement reconstruction period; and (iv) commemorative betterment period. Such an event can impact the society the impact may be related to the industry output causing shortage of goods and services that in turn may directly affect the health and welfare of people. Moreover, the recovery process is related to the level of society readiness. Quarantelli and Dynes (1976) and Hanchett et al. (1998) also explored the social impact of an extreme event but they explained social impacts in terms of physical and economic losses. It seems it is common to simultaneously describe the event consequential losses by both those impacts. For example, Grunfest (1995) counted the losses of an event as the number of destroyed houses or businesses, tax losses, and migration from an affected area. In this manner, the physical and social impacts can be known as inherent factors.

As noted above, the loss associated with a disaster is usually difficult and sometimes impossible to measure. The combination of two such as physical and social impacts makes the evaluation even more complex. Therefore, the social impact is assumed zero and the behaviour of individuals, families, and communities are supposed to be constant and as pre-disaster behaviour (but the social loss is subsumed to the economic loss in this case).

Psychological impact

Green *et al.*, (1990) noted that people who are directly affected by an extreme event can suffer psychiatric disorders after the event much more than those indirectly affected. These disorders are called psychological impact of an event. Galea *et al.* (2002) stated that disasters, extensive loss of life, damages to buildings and properties, and financial strain appear to be the main factors that cause psychological impacts. However, Burdge (1995) suggested the psychological impact of an event should be a part of the social impact. The psychological impact of an event can affect and change the behaviour of people. It was argued that short term and long lasting disorders change human consumption behaviour (demand). For example, after an extreme event, the demand for phone calls may significantly increase yet the communication infrastructure that operates, is still based on its pre-disaster capacity; a higher than normal demand can theoretically *shut down* the whole communication system.

In conclusion, there are rather close relationships between psychological and physical and social impacts. The psychological impact may be also defined by physical impact in an economic assessment when it is determined via the changes in consumers demand. Therefore, a comprehensive loss assessment needs to be carried out through these three impacts but the complexity of the mixture of the above mentioned impacts is rather high and would lead significant modelling difficulties with little improvement in accuracy it would seem when a first level attempt is made to model loss in economy. Hence, to avoid the complexity, the psychological impact is assumed neutral (subsumed in the total economic loss).

Development of a new model based on the input output model (I/O)

The dynamic input - output model has been developed and described by Miler and Blair (1985). It combines the relationship between the industries (as systems), and also provides an analytical technique to study the whole system (an economy) during the recovery process. The new model analyses events within the same framework but an important novel factor is then included; that is, the severity of the event based on the extreme value theory. In traditional dynamic I/O model the recovery of an affected industry starts at time zero after the disaster with a constant resilience rate; increasing to reach its as-planned output. This process is the same for any event regardless of magnitude. However, the magnitude of an event may be different for each industry. Therefore, the I/O model can only be used for a basic level of disaster (when $n = 1$). In this paper, the model is revised based on probability theory and extreme value distribution concepts. Evaluating the industry outputs associated with the event size, either independently or collectively, needs be done for all affected important industries (infrastructure systems). The important industries like infrastructure systems are the key industries for the stability of supply and demand chain in an economy. Since the magnitude of an event is also considered, the new model will provide a more accurate measurement of economic loss when compare to other models.

However, for this purpose the output reduction and/or outage must be shown clearly. For instance, it is assumed that a powerhouse (as an important industry) is affected by an extreme event and results in several hours or days of electricity outage. There is no damage on electricity network and that the electricity consumer industries do not have backup generators. Importantly, there is only one power house producing electricity for all. This means the source of electricity is the same for all industries, and end-users such as the local people in the selected economic environment (this means that all face an equal risk of electricity outage).

The term *an important industry* can also contain services industry as long as they behave the same as manufacturing industries in pre- and post-disaster. Shobeiri Nejad and Tularam (2010) formulated the Australian international arrivals and argued that tourist arrivals, as a variable, follows product life cycle similar to other industries. Tularam *et al.* (2012) tested this formula over a time series analysis and the result was enough close with 95% confidence to validate that. This variable plays a significant role in a tourist destination while other industries and enterprises are tightly connected to tourism industry. Therefore, tourism industry may also count as an important industry that have outage (no tourist arriving) or reduce and regrow over the recovery process.

Model development based on probability and extreme value theories

Consider a random variable Z (population) with a known probability distribution $f_Z(z)$ (PDF: Probability Distribution Function) or $F_Z(z)$ (CDF: Cumulative Distribution Function). The population is assumed to have a size n . Any sample of some size drawn from the population contains a largest and smallest value. Under repeated sampling, these extreme values will have their own distribution related to initial population (Z).

The sample size n contains a set of observations. This set includes the first, second, up to the n^{th} observation. While these observations are unpredictable, they may be accepted as real values of random initial variables (Z_1, \dots, Z_n). The following sets (U_1, U_n) may be derived for minimum and maximum random initial variables: $U_1 = \min(Z_1, \dots, Z_n)$ and $U_n = \max(Z_1, \dots, Z_n)$

The above sets include the extreme values that in turn will form their own distributions. These probability distributions may be extracted under certain assumptions when sampled repeatedly. For example, it is assumed that maximum set of random initial variable (U_n) is less or equal to a specific value for example: u . Therefore, all initial random variables are less than or equal to u ($Z_1, \dots, Z_n \leq u$). If it is assumed that all initial random variables are statistically independent then the distribution functions of initial variables (Z) will be equal: $F_{Z_1}(z) = \dots = F_{Z_n}(z) = F_Z(z)$ and based on this assumption, the CDF of U_n may be written as shown in Equation 1:

$$F_{U_n}(u) = P(Z_1 \leq u, Z_2 \leq u, \dots, Z_n \leq u) = [F_Z(u)]^n. \quad 1$$

The PDF is the derivative of CDF and may be written as shown in Equation 2:

$$f_{U_n}(u) = F'_p(y) = \frac{dF_{U_n}(u)}{du} = n [F_Z(u)]^{n-1} f_Z(u). \quad 2$$

For the set of min (minimum) random initial variables (U_1), it is assumed that U_1 is larger than a specific number u . Therefore, all initial random variables are larger than u ($Z_1, \dots, Z_n > u$) and the distribution of U_1 may be written as shown in Equation 3:

$$1 - F_1(u) = P(Z_1 > u, Z_2 > u, \dots, Z_n > u) = [1 - F_Z(u)]^n. \quad 3$$

Equation 5.3 is also known as the *survival function* (Ang and Tang, 2007). The CDF of this can be given as:

$$F_{U_1}(u) = 1 - [1 - F_Z(u)]^n. \quad 4$$

Therefore, the PDF is again the derivative of Equation 4:

$$f_{U_1}(u) = n [1 - F_Z(u)]^{n-1} f_Z(u) \quad 5$$

Equation 3 is survival function and approaches zero, thus $F_{U_n}(u)$ is chosen for probability distribution in this model development. The initial variable Z may be assumed to be the population of events and $F_{U_n}(u)$ represents the probability distribution of events. It is logical to accept that where there is no event, there is no sample size and the probability of event is zero (vice versa); on the other hand, where there are several events the random variable may contain the set of maximum (U_n) and minimum: U_1 . An event was defined to be a disaster with numerical outcomes and, respectively, it can be explained by the consequential damages ($event \equiv damage$). Hence, the initial variable can be substituted by population of damage. This idea can be also expanded for the event size. When the event size reaches and then exceeds a specific value, the population of damage becomes (and remains) 1. It is also true that the larger the size of event the higher the size of damages. Therefore, a sample from initial variable may contain n different level of damages associated with the events with different sizes.

Although the damage may be measured in various ways, for the purpose of this research it is measured by the degradation of the industry output. Therefore, the initial damage population is substituted by the degradation of affected industry outputs. This amount of output can be also normalized and oscillate between 0 and 1. The recovery process and damage are two sides of the same coin. While the damage degrades the system outputs, the recovery process aims to return that to pre-disaster value ($recover \equiv \frac{1}{damage}$). Therefore, the initial variable in extreme value theory may be replaced by normalized output over the recovery process. It is a fact that each industry is built based on its environmental conditions where all the catastrophic events were possibly considered (there is an ideal zero events situation). However, environmental conditions change over time and, subsequently, the probability of uncertain and unforeseen events increases. As a result, the probability of damage on the industry rises. The level of damage on the industry may be also enlarged by increasing severity of the events. The damage may then directly affect the industry output and degrade it. Following the industry output degradation, the recovery processes become activated and attempts are made fix the system so that the output may return to its pre-disaster (so-called as-planned) output. This recovery is a dynamic process. During this process, the system outputs (as a measurable numerical outcome) increases over the time of recovery. This numerical outcome can be accepted to be explained by time (variable). If each initial variable is shown by corresponding time T , U_n may be shown by $U_n = \max(T_1, \dots, T_n)$. Based on above modification of extreme value theory and statistical content, $F_{U_n}(u)$ models the amount of output over the recovery process. Equation 5.1 may be written as shown in Equation 6:

$$F_{U_n}(t) = [F_T(t)]^n \quad \text{where} \quad (T_1, \dots, T_n < t). \quad 6$$

Woo (2000) proposed that Natural disasters such as earthquakes and volcanic eruptions follow a random *poisson* distribution in time. Embrechts *et al.* (2003) noticed that *poisson* process represents rare events in space and time. This process encounters in the context of the risk theory and is a key tool in analysis of external events. Ang and Tang (2007) noted that the occurrence of most extreme natural events approximately follow a *poisson* process. The *poisson* process is based on the occurrence time itself. If the occurrence of an extreme event follows a *poisson* process, then the time and recurrence may be defined by the exponential distribution. Therefore, $F_T(t)$ may be written as shown in Equation 7:

$$F_T(t) = 1 - e^{-kt} \quad 7$$

In Equation 5.7, notation k refers to *mean of occurrence rate*. In substituting the probability of event occurrence with the system output through above process, the notation k can now be redefined as *the mean growth rate*. The exponential growth is dynamic and based on two variables: namely, time and growth rate. This model is stable for all values of these two variables. Indeed, the exponential growth model has been used in studies by Lian (2006) and Santos (2006), both of whom demonstrate the system resilience after an event. The exponential growth also satisfies the goal of recovery process (the industry output can regain its pre-disaster value (to the as-planned output)).

Based on Equation 1 and Equation 7, the distribution of U_n may be written as followed:

$$F_{U_n}(t) = [1 - e^{-kt}]^n. \quad 8$$

Equation 8 is the amount of the system output (normalised) over recovery time t . $F_{U_n}(t)$ represents the normalised system output ($\frac{x(t)}{x}$) (Soomro, Tularam et al, 2013). Now Equation 7 can be now written as shown in Equation 9:

$$\frac{x(t)}{x} = [1 - e^{-kt}]^n, \quad 9$$

where, $x(t)$ refers to the system output or function, x shows the as-planned output, t refers to time ($t \geq 0$), k directs to the constant growth rate (*recovery resilience rate and $k > 0$*); and n the magnitude of the event ($n \geq 1$). The PDF of the speed of system output growth over the recovery process is $f_{U_n}(t) = n k e^{-kt} [1 - e^{-kt}]^{n-1}$. This model, as it mentioned earlier, is more realistic for the recovery after an extreme event because of embedding the event size n . The previous recovery model was developed and used for all events regardless to their size. However, in new model, when the event size increases, the effects are visible and measurable. Equation 9 can be expanded via the binomial theorem for small n :

$$\begin{aligned} \frac{x(t)}{x} &= [1 - e^{-kt}]^n = 1 + \frac{n!}{1!(n-1)!} (-e^{-kt})^1 + \frac{n!}{2!(n-2)!} (-e^{-kt})^2 + \frac{n!}{3!(n-3)!} (-e^{-kt})^3 + \dots + (-e^{-kt})^n \\ &= 1 + n(-e^{-kt})^1 + \frac{n(n-1)}{2!} (-e^{-kt})^2 + \frac{n(n-1)(n-2)}{3!} (-e^{-kt})^3 + \dots + (-e^{-kt})^n. \end{aligned} \quad 10$$

Equation 10 is useful for small n .

Equation 9 can also be expanded based on exponential function theory for large n . Based on this theory $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)$ can be written as e^x . Therefore, by substitution of x with $-ne^{-kt}$:

$$\frac{x(t)}{x} = \lim_{n \rightarrow \infty} [1 - e^{-kt}]^n = \lim_{n \rightarrow \infty} \left[1 + \frac{(-ne^{-kt})}{n}\right]^n = e^{-ne^{-kt}} \quad 11$$

Before developing the model for the interconnections between the systems, a shifted exponential distribution needs to be investigated. The recovery process, while the post disaster function of system has dropped, can be explained by shifted model. In chapter 2, this shift was explained by adding as-planned output (x) and the output after the disaster (x_0). However, this shifting is explaining by t_0 in this model; that is, t_0 is the corresponding time for x_0 . In equation 12, n has the lowest value ($n=1$).

$$\frac{x(t)}{x} = 1 - e^{-k(t-t_0)}. \quad 12$$

Rearranging Equation 12 in term of t_0 is:

$$t_0 = \frac{1}{k} \ln \left(\frac{x-x_0}{x} \right) \quad 13$$

In a similar manner for large n , the above method can be used to determine the extreme CDF model for Equation 11:

$$\frac{x(t)}{x} = e^{-ne^{-k(t-t_0)}} \quad 14$$

Therefore, t_0 may be written as shown in Equation 15:

$$t_0 = \frac{1}{k} \ln \left(\frac{\ln \left(\frac{x_0}{x} \right)}{-n} \right). \quad 15$$

As in real situation n is limited, equation 15 can be accepted to estimate the time. Later, in this research, Equation 9 is hired to show these relationships by the redefinition of recovery time and outage time.

Figure 5 shows the distribution of the system output during the recovery process. That is, when x_0 is larger than zero, the curve is shifted more to the left side.

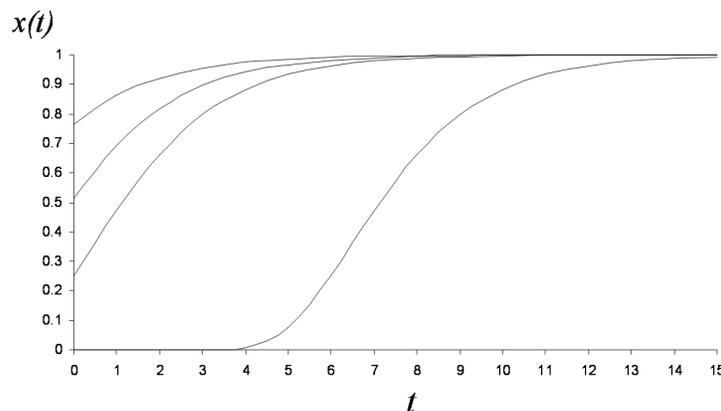


Figure 5: System output over the recovery process

Figure 5 demonstrates the curves for four different x_0 values. When the output degrades to zero, it may halt for a portion of time over the recovery process. However, when x_0 is greater than zero, the halting time may be shorter or zero. Moreover, Figure 5 shows that in some cases when the output is larger than zero, the growth starts without any halting time over the recovery process.

In above method, the model was shifted by a calculated or estimating the initial time. The model can be also shifted by altering other parameters. In the following innovative method, the model is shifted by modifying the event size n . The solution of original distribution associated with x_0 , instead of t_0 , can be given as:

$$\frac{x_0}{x} = 1 - \left(1 - \frac{x_0}{x}\right)e^{-kt} . \quad 16$$

Equation 16 can be written rearranged to the following form:

$$\frac{x_0}{x} = x_0 + \left(1 - \frac{x_0}{x}\right)(1 - e^{-kt}) . \quad 17$$

Assuming the effect of an event with larger size causes more halting time, Equation 17 can be written as:

$$\frac{x_0}{x} = x_0 + \left(1 - \frac{x_0}{x}\right)(1 - e^{-kt})^n , \quad 18$$

where the size of event imbedded in the parameter n . The event size in Equation 18 was explained earlier when the extreme value theory was discuss and presented in Equation 8. The recovery process is shown by following curves (Figure 6):

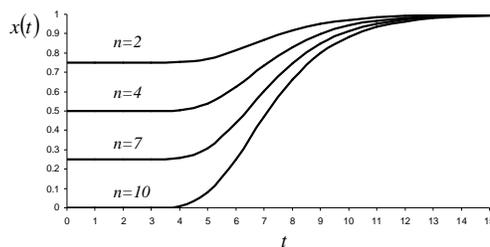


Figure 6: Equal halting time in the beginning of recovery process

Figure 6 shows that the halting time is the same for any value of x_0 . To make it realistic, it is assumed that event size (n) explains the halting time of recovery when the system is affected. A shift in the event size n allows this to be represented. It is also assumed that when an event occurs with magnitude, n ; while the system is not completely damaged (and has output more than zero), the halting time is shorter than when it is completely destroyed. In this case, the event size needs to be altered and this results in an amended halting time (Figure 7). Examining the relationship between natural disasters and event size (Figure 5.3), the damage power of an event to may grow exponentially (such as in an earthquake) as modelled in Equation 19.

$$Damage = (1 - e^{-\lambda n})^\nu \quad 19$$

For example, for $\nu = 5$, $\lambda = 0.3$ and $n = 8$, the corresponding damage is 62%; that is, when $x_0 = 1 - 0.62 = 0.38$.

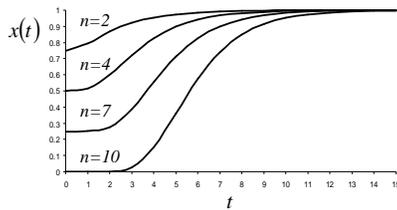


Figure 7: Amended halting time in the beginning of recovery process

Miller and Blair (1985) proposed a well-known dynamic balance equation for Input-Output model. Theoretically, this model is based on basic population growth model as shown in Equation 20;

$$X(t) = A X(t) + Y(t) + B \dot{X}(t) \quad \text{or}$$

$$\dot{X}(t) = \frac{I}{B} [(I - A)X(t) - Y(t)], \quad 20$$

where the label A represents a matrix of technical coefficients, Y is a matrix of final demand, and B is a diagonal capital matrix. It is to be noted that as the positive value of B increases the industry outputs $X(t)$ to infinity. Therefore, Equation 20 is stable when the capital coefficient matrix has negative values (adapted from Miller and Blair, 1985). If B accepted as matrix with negative values, $\frac{I}{B}$ is also accepted as the same.

However, recovery is a positive and additive process. Therefore, the coefficient $\frac{I}{B}$ is substituted by matrix by $-K$. Matrix K has positive values (the resilience matrix) and contains the growth rate for each industry. The negative sign attached to K can be explained by the positive and additive inherent nature of resilience or recovery in the exponential solution.

$$\dot{X}(t) = -K(I - A)X(t) \quad 21$$

$$\text{or } \frac{dX(t)}{dt} = -K(I - A)X(t) \Rightarrow \frac{dX(t)}{X(t)} = -K(I - A)dt \Rightarrow$$

$$X(t) = X_0 e^{-K(I-A)t}$$

This solution is a descending function (survival function) and can not show the recovery process. However, if $X(t)$ and X_0 can be substituted by $X - X(t)$ and $X - X_0$ respectively, the solution can represent the recovery process. This means that the degradation of the system outputs follows a survival function and at the end of recovery process, the systems obtain their pre-disaster (as-planned) outputs (Equation 22).

$$X(t) = X_0 + (I - e^{-K(I-A)t})(X - X_0) \quad 22$$

For an event size (n), Equation 22 can be written as:

$$X(t) = X_0 + (I - e^{-K(I-A)t})^n (X - X_0) \quad 23$$

where $(I - e^{-K(I-A)t})^n$ is exponential growth over recovery after an extreme event with size n (this equation relates to the extreme value, namely, severity of event). Equation 23 shows the output of many interdependent systems as a matrix. Equation 23 can now be exponentially expanded:

$$e^{-K(I-A)t} = (e^{K(I-A)t})^{-1} \quad 24$$

The right hand side (RHS) of Equation 24 (in the mean time without the negative power) when expanded gives:

$$e^{K(I-A)t} = \exp\left(\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \times \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) \times t\right) \quad 25$$

$$e^{K(I-A)t} = \exp\left(\begin{bmatrix} k_1(1-a_{11}) & -k_1a_{12} \\ -k_2a_{21} & k_2(1-a_{22}) \end{bmatrix} \times t\right) \quad 26$$

Finally, the RHS of Equation 24 may be written as:

$$\begin{bmatrix} e^{\left[\frac{k_1(1-a_{11})+k_2(1-a_{22})}{2}\right]t} \left(1 + \left[\frac{k_1(1-a_{11})-k_2(1-a_{22})}{2}\right]t\right) & -e^{\left[\frac{k_1(1-a_{11})+k_2(1-a_{22})}{2}\right]t} k_1a_{12}t \\ -e^{\left[\frac{k_1(1-a_{11})+k_2(1-a_{22})}{2}\right]t} k_2a_{21}t & e^{\left[\frac{k_1(1-a_{11})+k_2(1-a_{22})}{2}\right]t} \left(1 - \left[\frac{k_1(1-a_{11})-k_2(1-a_{22})}{2}\right]t\right) \end{bmatrix};27$$

When the negative power is added, the Equation 27 becomes:

$$e^{-K(I-A)t} = \left(\begin{bmatrix} e^{\left[\frac{k_1(1-a_{11})+k_2(1-a_{22})}{2}\right]t} \left(1 + \left[\frac{k_1(1-a_{11})-k_2(1-a_{22})}{2}\right]t\right) & -e^{\left[\frac{k_1(1-a_{11})+k_2(1-a_{22})}{2}\right]t} k_1a_{12}t \\ -e^{\left[\frac{k_1(1-a_{11})+k_2(1-a_{22})}{2}\right]t} k_2a_{21}t & e^{\left[\frac{k_1(1-a_{11})+k_2(1-a_{22})}{2}\right]t} \left(1 - \left[\frac{k_1(1-a_{11})-k_2(1-a_{22})}{2}\right]t\right) \end{bmatrix} \right)^{-1};28$$

Equation 28 may now be simplified to Equation 29:

$$\begin{aligned}
 & \left[\left(1 + \left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right]_t \right) \div \left(e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2} \right]_t} \left(1 - \left(\left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right]_t \right)^2 - (k_1 k_2 a_{12} a_{21} t^2) \right) \right) \right. \\
 = & \left[\left(k_2 a_{21} t \right) \div \left(e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2} \right]_t} \left(\left(1 - \left(\left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right]_t \right)^2 - (k_1 k_2 a_{12} a_{21} t^2) \right) \right) \right) \right. \\
 & \left. \left(k_1 a_{12} t \right) \div \left(e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2} \right]_t} \left(\left(1 - \left(\left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right]_t \right)^2 - (k_1 k_2 a_{12} a_{21} t^2) \right) \right) \right) \right. \\
 & \left. \left(1 - \left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right]_t \right) \div \left(e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2} \right]_t} \left(\left(1 - \left(\left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right]_t \right)^2 - (k_1 k_2 a_{12} a_{21} t^2) \right) \right) \right) \right]_{.29}
 \end{aligned}$$

Factorising the common coefficient, Equation 29 may be written as:

$$\begin{aligned}
 e^{-K(I-A)t} = & \frac{1}{e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2} \right]_t} \left(\left(1 - \left(\left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right]_t \right)^2 - (k_1 k_2 a_{12} a_{21} t^2) \right) \right)} \times \\
 & \left[\begin{array}{cc} \left(1 + \left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right]_t \right) & k_1 a_{12} t \\ k_2 a_{21} t & \left(1 - \left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right]_t \right) \end{array} \right]_{30}
 \end{aligned}$$

At this time other elements of the model may be added to Equation 30 in the following manner:

$$\begin{aligned}
 I - e^{-K(I-A)t} = & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2} \right]_t} \left(\left(1 - \left(\left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right]_t \right)^2 - (k_1 k_2 a_{12} a_{21} t^2) \right) \right)} \times \\
 & \left[\begin{array}{cc} \left(1 + \left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right]_t \right) & k_1 a_{12} t \\ k_2 a_{21} t & \left(1 - \left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right]_t \right) \end{array} \right]
 \end{aligned}$$

$$I - e^{-K(I-A)t} = \begin{bmatrix} 1 - \frac{\left(1 + \left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2}\right]t\right)}{e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2}\right]t} \left(1 - \left(\left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2}\right]t\right)^2 - (k_1 k_2 a_{12} a_{21} t^2)\right)} \\ \frac{-k_2 a_{21} t}{e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2}\right]t} \left(1 - \left(\left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2}\right]t\right)^2 - (k_1 k_2 a_{12} a_{21} t^2)\right)} \\ \frac{-k_1 a_{12} t}{e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2}\right]t} \left(1 - \left(\left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2}\right]t\right)^2 - (k_1 k_2 a_{12} a_{21} t^2)\right)} \\ 1 - \frac{\left(1 - \left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2}\right]t\right)}{e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2}\right]t} \left(1 - \left(\left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2}\right]t\right)^2 - (k_1 k_2 a_{12} a_{21} t^2)\right)} \end{bmatrix} \quad 31$$

The equation $X(t) = X_0 + (I - e^{-K(I-A)t})(X - X_0)$ may now be written as shown in Equation 32:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 1 - \frac{\left(1 + \left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2}\right]t\right)}{e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2}\right]t} \left(1 - \left(\left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2}\right]t\right)^2 - (k_1 k_2 a_{12} a_{21} t^2)\right)} \\ \frac{-k_2 a_{21} t}{e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2}\right]t} \left(1 - \left(\left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2}\right]t\right)^2 - (k_1 k_2 a_{12} a_{21} t^2)\right)} \\ \frac{-k_1 a_{12} t}{e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2}\right]t} \left(1 - \left(\left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2}\right]t\right)^2 - (k_1 k_2 a_{12} a_{21} t^2)\right)} \\ 1 - \frac{\left(1 - \left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2}\right]t\right)}{e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2}\right]t} \left(1 - \left(\left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2}\right]t\right)^2 - (k_1 k_2 a_{12} a_{21} t^2)\right)} \end{bmatrix} \times \begin{bmatrix} x_1 - x_1(0) \\ x_2 - x_2(0) \end{bmatrix} \quad 32$$

The event size n is noted to be one. Therefore, for every interconnected system (in a system), the output over the time can be written as:

$$x_1(t) = x_1(0) + \left[1 - \frac{\left(1 + \left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right] t \right)}{e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2} \right] t} \left(\left(1 - \left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right] t \right)^2 - (k_1 k_2 a_{12} a_{21} t^2) \right)} \right] \times (x_1 - x_1(0))$$

$$+ \left[\frac{-k_2 a_{21} t}{e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2} \right] t} \left(\left(1 - \left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right] t \right)^2 - (k_1 k_2 a_{12} a_{21} t^2) \right)} \right] \times (x_2 - x_2(0))$$

$$x_2(t) = x_2(0) + \left[\frac{-k_2 a_{21} t}{e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2} \right] t} \left(\left(1 - \left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right] t \right)^2 - (k_1 k_2 a_{12} a_{21} t^2) \right)} \right] \times (x_1 - x_1(0))$$

$$+ \left[1 - \frac{\left(1 - \left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right] t \right)}{e^{\left[\frac{k_1(1-a_{11}) + k_2(1-a_{22})}{2} \right] t} \left(\left(1 - \left[\frac{k_1(1-a_{11}) - k_2(1-a_{22})}{2} \right] t \right)^2 - (k_1 k_2 a_{12} a_{21} t^2) \right)} \right] \times (x_2 - x_2(0))$$

33

As mentioned previously, the event may affect more than one industry at the same time. The event size may also be different for each industry as well. In this case, the event size needs to be provided as a diagonal matrix. The result is an exponential matrix power to a matrix. At the present to avoid this interfering of event sizes and complexity, it is assumed that the event simply impacts one industry directly. This assumption simplifies the case. Therefore, there is just one event size (n) taken to be a real number. This process is repeated for all affected industry individually, and finally, the smallest output for each time is accepted.

To illustrate the application of the newly developed I/O model, the following example is selected and modified from Miller and Blair (1985). In fact, Lian (2006) also used this example to demonstrate IIM and DIIM models (See Table 1). Table 2 is modified for a one industry economy.

Table 1: Flow of goods among two industries economy (million dollars)

Industry	<i>I</i>	<i>j</i>	<i>Final Demand</i>	<i>Total Output</i>
<i>i</i>	150	500	350	1000
<i>j</i>	200	100	1700	2000
<i>Primary Input</i>	650	1400		
<i>Total Input</i>	1000	2000		

Table 2: Flow of goods among one industries economy (million dollars)

Industry	<i>i</i>	<i>Final Demand</i>	<i>Total Output</i>
<i>i</i>	150	850	1000
<i>Primary Input</i>	850		
<i>Total Input</i>	1000		

Figure 9 represents the outputs of two industries through a complete recovery process concerning industry *i*. This industry is affected by an extreme event with magnitude of one. The halting time for industry *i* is zero. The recovery resilience rate is constant and equal 0.2 ($k=0.2$). Figure 8 shows that although industry *j* is unaffected, its output is decreasing and then increasing to as-planned output while industry *i* is under recovery.

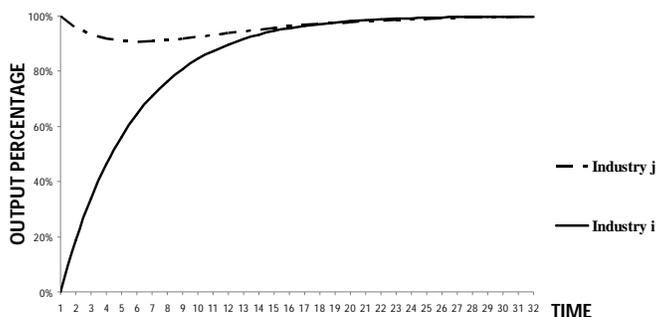


Figure 8: Normalised output through a complete recovery process on industry *i*

Increasing n changes the curve into double exponentials. There is an observed halting at the beginning of the recovery process. The halting can be defined by as-planned output of affected industry. It can be assumed that the industry with the output less than 95% of as-planned to be an extinguished industry. Figure 10 demonstrates the output of two industries for different event magnitudes.

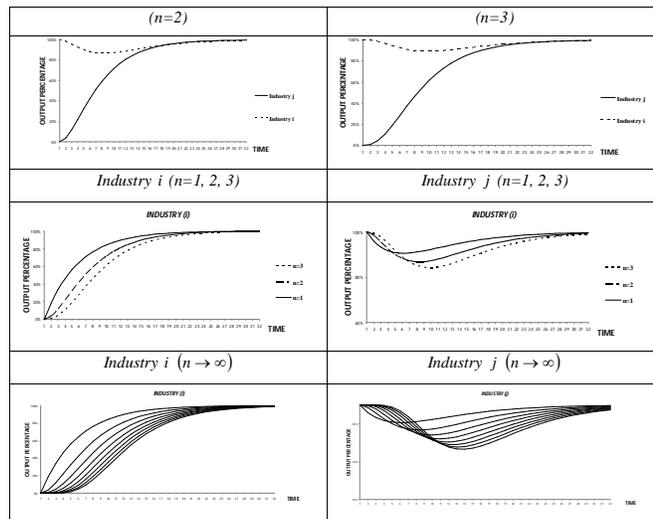


Figure 9: Industry outputs through recovery process with different event magnitudes

Figure 9 shows that as the magnitude of event increases, the halting time also increases for the affected industry. On the other hand, unaffected industries, respectively, decrease with a pause associated with halting time of affected industry. However, an affected industry also impacts other industries through the interconnected network (at the same time). For this reason, the function is changed to a Min function. This Min function selects the smallest value of the industry outputs, while the event size increases. For affected industry the Min function and the previous one have the same outcomes, however, the previous function has a weakness to calculate the industry output which is affected through another industry. This weakness appears when the directly affected industry has a halting time by increasing the event size. In this case, the interconnected industry experiences the same halting time but in an opposite way, it stays without any changes. Realistically, the industry outputs must be smaller while the event size is larger. Therefore, Min function solves this problem makes the outcomes real.

$$X(t) = \text{MIN} \left\{ X_0 + \left(I - e^{-K(I-A)t} \right)^n (X - X_0) \right\} \text{ when } n = 1, 2, 3, \dots \quad 34$$

Figure 10 demonstrates the amended graph of normalized output for two industries during recovery process with event size of 8 ($n=8$).

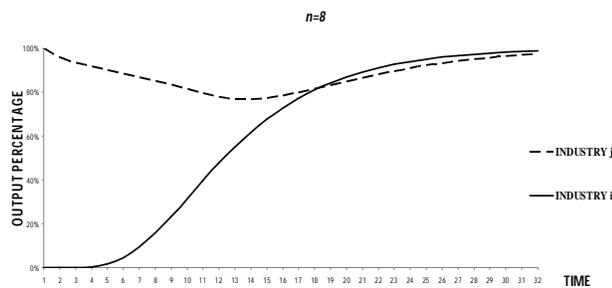


Figure 10: The graph of normalised output through recovery process with *Min* function

Estimating the parameters

The information regarding the matrix A that shows the monetary interdependency between the industries can be obtained from ABS website (Australian Bureau of Statistics-Australian National Accounts (Input-Output Tables-Industry by Industry Flow Table). Other related data is also available for 406 detailed US industries from US Department of Commerce (BEA/RIMS II). The information regarding the matrix X , which shows as-planned output, can also be obtained from Input Output table provided in the ABS website (www.abs.gov.au). The information regarding the matrix X_0 that represents the output after an extreme event, can be computed through simulating some *what-If* scenarios, or gained from experts and decision makers interested in evaluating the risk of events. There are two parameters to estimate namely, k and n . The estimation must be through some of simulated condition by experts, fully explained foundations, or through an estimation process by studying the historical data. As the output of each industry is related to the event size (n) and recovery resilience rate (k), an estimation of event size may be based on identifying the response of industries while there are facing with new events as well as maximum possible magnitude of new unknown events. The resilience rate may be also estimated by the amount money that is spent for restoration, human recourse, and management.

Estimating event size

The event size is taken as natural number ($n \in N^+$), N^+ (where zero is not included). The value of zero for n shows the absence of any event. In this case, the variable n is categorized from *Not-Bad* to *Very-Bad*. After an event with *Not-Bad* event size, n , the output may drop to zero (For example: electricity outage). However, within in a short time recovery process, the output increases to 95% or more of as-planned output. On the other hand, an event with *Very-Bad* size n is assumed to result in the industry completely destroyed. In this situation, although the recovery process starts, the outage time (less than 95%) is assumed to be lengthy. The event size does not have upper limit and there is a possible propensity to infinity. However, the maximum value for n in this case can be interpreted by an outage for one financial year or the degraded output does not have any growth. Based on above explanation, the event size is starting with a lower boundary of one and limited by an arbitrary decision for the upper boundary. In engineering modelling applications such in Dam design, n is chosen as number of years for designing of *Dam* as the lifetime of the Dam. In Multi-hazard Loss Estimation Methodology for Earthquake Model (HAZUS-MH MR3), the damages of event are categorized into four levels: Minor, Moderate, Extensive, and Complete (Federal

Emergency Management Agency, 2003). Based on these categories and applications (and for simplicity as well), the event size is selected to be up to a maximum of 10 (industry is completely destroyed). If the event size is also assumed as random Poisson distribution, then the above values of n can be described as the mean of distributed size.

Estimating recovery resilience rate and outage time

The system outputs are explained by three parameters: event size (n), recovery resilience rate (k), and the time outage or recovery ($T_{outage}, T_{recovery}$) (Equation 5.23). Each of those parameters may be estimated or calculated using the other two parameters. For this purpose, following three innovative methods are used in this thesis.

Method 1: The outage (or halting) time may be obtained or assumed from

- (i) historical data;
- (ii) by comparing the outage time with other similar event; or
- (iii) as suggested by expert.

When the event size is known, the recovery resilience rate and standard recovery time can be determined using Equations 35 and 36.

$$k = \left(Ln \left[1 - (0.05)^{(1/n)} \right] \right) / -T_{outage} \quad 35$$

$$T_{Recovery} = \left(Ln \left[1 - (0.95)^{(1/n)} \right] \right) / -k \quad 36$$

Both of above equations have been derived from Equation 23. The outage (or halting) time is defined as a part of recovery time that the output is less than or equal to 5% of pre-disaster output. For example, if the output drops to zero after the event and the recovery process starts immediately (initial time is zero) and the corresponding standard outage time of 5% of as-planned (or 0.05 if the as-planned is normalised) is given, the recovery resilience rate can be calculated using Equation 35. Similarly, the standard recovery time (when the recovery process has completed and the output is equal to or more than 95% of as-planned output or 0.95 of the normalised output) can be obtained using Equation 36. In fact, in this method, the resilience rate is determined based on the outage time T_{outage} ; and using the now known resilience rate, the recovery time $T_{Recovery}$ can then be calculated as well. Table 3 shows as example of this method of calculation.

The outage time, associated with event size, may be also described using an exponential distribution. Table 3 shows the calculated parameters for four different event sizes. The distribution of outage time may be described using a logarithmic scale or Weibull distribution as well (Figure 11). The distribution can be selected based on the historical extreme values distribution. In the absence of historical information, the distribution may be chosen by explaining the case. For example, it can be said that, by increasing the event size more industries are affected (positive loop). On the other hand, the number of industries is limited (negative loop). Both these loops and their combination may be defined by exponential distributions.

Table 3: Parameter estimation by assumed outage time and event size- exponential base

Event Size	Magnitude n	Standard Outage Time $x_t \leq 0.05 x$	Standard Recovery Time $x_t \geq 0.95 x$	Associated k
No event	0	N.A	N.A	N.A
Minor	=1	<i>Less than one day e.g. about 1 hour</i>	3 days	1
⋮				
Moderate	=4	1 day	7 days	0.64
⋮				
Extensive	=7	14 days	65 days	0.075
⋮				
Complete	≥ 10	100 days	390 days	0.0135

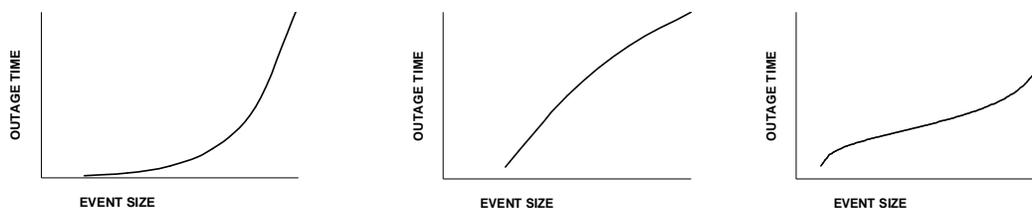


Figure 11: Relationships between outage time and event size

By assuming the same flooring and sealing for outage time, Table4 shows the estimated parameters for exponential outage distribution with maximum value. It can be argued that if an industry completely destroyed by an assumed event with size 10, any larger events cannot result in more damage and outage time. Finally, the distribution of outage may follow a combination of these distributions.

Table 4: Parameter estimation by assumed outage time and event size-
Logarithmic Base

Event Size	Magnitude n	Standard Outage Time $x_t \leq 0.05 x$	Standard Recovery Time $x_t \geq 0.95 x$	Associated k
No event	0	N.A	N.A	N.A
Minor	=1	<i>Less than one day e.g. about 1 hour</i>	3 days	1
⋮				
Moderate	=4	45 days	307 days	0.0142
⋮				
Extensive	=7	80 days	373 days	0.0132
⋮				
Complete	≥ 10	100 days	390 days	0.0106

The Tables 3 and 4, show that that *the recovery resilience rate* for each industry falls clearly between an interval; 0 and 1 ($0 < k \leq 1$). This interval is useful to demonstrate the maximum possible resilience rate based on available resources to the decision makers. Also, when k approaches to zero the recovery is rather slow; and both the outage and recovery are increasing.

Method 2: This method is similar to method 1 but the standard recovery time, T_{Recovery} , is used instead of the outage time. In this case the recovery time (the output returns to 95% as-planned output) is known or it is decided. The resilience rate and standard outage time can be determined using Equations 37 and 38.

$$k = \left(\text{Ln} \left[1 - (0.95)^{(1/n)} \right] \right) / -T_{\text{Recovery}} \quad 37$$

$$T_{\text{outage}} = \left(\text{Ln} \left[1 - (0.05)^{(1/n)} \right] \right) / -k \quad 38$$

Method 3: In this method, it is assumed that k is constant for any value of n and the recovery resilience rate

is larger than zero with an upper boundary of 1. The outage time and recovery time are dependent to k . The Tables 5.5 and 5.6 show the outage and recovery times for different event sizes.

Table 5: Outage and recovery time for different event sizes ($k = 0.0106$)

$k = 0.0106$		
	$T_{Outage} / days$	$T_{Recovery} / days$
$n = 1$	5	285
$n = 4$	63	414
$n = 7$	102	467
$n = 10$	130	500

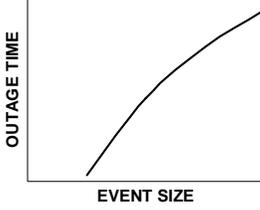
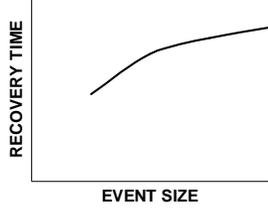



Table 5 shows that if $k = 0.0106$ (as an example), the recovery process will take 285 days to increase the output from 0 to 95% as-planned for lowest value of n ($n = 1$). However, this process takes 500 day for maximum value of n ($n = 10$).

Table 6: Outage and recovery time for different event sizes ($k = 1$)

$k = 1$		
	$T_{Outage} / days$	$T_{Recovery} / days$
$n = 1$	0.05	3
$n = 4$	0.7	4.5
$n = 7$	1.06	4.9
$n = 10$	1.36	5.3

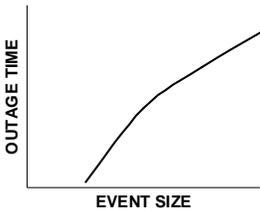
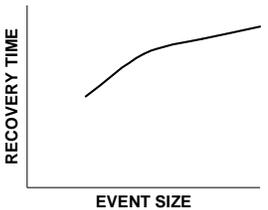



Table 5.6 shows that when resilience rate k is one for event size one the recovery time takes 3 days (increasing the output from 0 to 95% as-planned). This recovery process takes just 6 days for event size 10. The small difference between the recovery time for minimum and maximum events may be used to show that this method is not realistic. For example, the difference between recovery times for an industry in two conditions of slightly damaged and completely destroyed cases is a short 3 days (the resilience rate is 1 for both cases).

Model stability, limitation, and application

Stability

Equation 23 demonstrates the monetary flow through a dynamic recovery process. It is assumed that recovery resilience rate and event size are constant. Therefore, at the initial time ($t = 0$):

$$X(0) = X_0 + (I - e^{-K(I-A)*0})^n (X - X_0) = X_0 + (I - e^0)^n (X - X_0) = X_0. \quad 39$$

As the time progresses ($t \rightarrow \infty$), the degraded output increases from initial value to as-planned output (Equation 40).

$$X(\infty) = X_0 + (I - e^{-K(I-A)*\infty})^n (X - X_0) = X_0 + X - X_0 = X \quad 40$$

The case of the recovery resilience rate being positive ($k > 0$) affects the stability of Equation 23. For event size zero, this equation can be written as:

$$X(t) = X_0 + (I - e^{-K(I-A)t})^0 (X - X_0) = X \quad 41$$

All the members of matrix $I - e^{-K(I-A)t}$ are less than one, therefore, all the members of matrix $(I - e^{-K(I-A)t})^n$, when event size n increases ($n \rightarrow \infty$), approach to zero and therefore, the Equation 23 remains stable (Equation 42).

$$X(t) = X_0 + (I - e^{-K(I-A)t})^n (X - X_0) = X_0 \quad 42$$

Limitations

There is a possibility that an event impacts more than one industry at the same time. The same event can impact different industries by quite different sizes and outage times at the same time. However, the event size n in the new model can accept just one value as mentioned earlier. To solve this limitation, the impact of an event on each system and the corresponding cascade effect on other interconnected systems are calculated individually. This means that to evaluate the loss by this model, just one industry is accepted to be directly affected by the event at each time. This process is sequentially repeated for all directly affected industries with their associated event sizes. The effects of an event on an industry can be induced on other interconnected industries at the initial time. If there were no induced initial effects on the rest of interconnected industries, matrix of $(X - X_0)$ has one non-zero member that is related to the directly affected industry. Otherwise, any observed induced initial affects must be mentioned in the matrix of initial outputs (X_0). Hence, matrix $(X - X_0)$ may have more than one non-zero member. This process is conducted for each affected industry and interconnected industries individually. This procedure may lead to different values of outputs for each industry. In the final step, the minimum output in each time and for each industry is accepted.

Application

One basic application of this model is the measurement of the loss. In this application, if the event size and recovery resilience rate and/or outage time are either known or decided, the total loss of each industry as well as whole interconnected industries can be calculated over the time of recovery. There is also an opportunity for decision makers to minimize the cost and/or loss by optimizing the recovery process. The calculated loss provides the opportunity to decrease the event size in future by empowering important industries (critical infrastructure systems) through a protection manifest before the event happens and also speeding the recovery process by increasing the level of readiness.

Alternative model (demand based model)

Industries within an economic system can be classified in two major groups based on their role: supply industry or demand industry. Chen (2006) describes the classifications by two methods. The first method is classified based on the number of connections. In this method the number of industries that one specific industry buys from (input) or sells to (output) is counted and shown in two different columns. If the number of buyers is more than sellers, then the industry is called *Supply Industry* (else *Demand Industry*). The second method of classification compares the total dollar (as currency) income of sale and total dollar outcome of purchase for each industry from intermediate transaction in the I/O table. If the value of sale was more than purchase in one particular industry, the industry is *Supply Industry* (or else *Demand Industry*).

In Equation 23, the technical coefficient matrix A can be substituted by another diagonal matrix. If the technical coefficient matrix contains the coefficient based on the amount of output from one industry to another (supply-side), the substituted diagonal matrix is based on the input to one industry from another (demand-side). These two models may be then used based on the definition of industries. To evaluate the loss via one affected supply industry, Equation 23 may be applied. However, if the industry is a demand industry, the loss may be measured through the alternative model where matrix A is substituted by an alternative matrix (See Ghosh, 1958)

Event size distribution and calculation of total loss

The important industries (critical infrastructure systems) are specifically built based on many factors including historical climate, current climate, and essential security information of the location. Therefore, the probability and possibility of concurrency of disastrous events with larger sizes may increase by changing each one of the factors. The only parameter that accepts randomness in this thesis is the event size. To randomize the event size, the *truncated Poisson distribution* might be selected. The truncated Poisson distribution is a Poisson distribution without value of zero that shows the absence of the event (*No-event*). The loss may be obtained by adding different categories of losses and costs namely: loss through

- perturbation in monetary flow among the industries: this loss can be estimated and explained by developed model;
- changing the behaviour of end-users e.g. local people: this loss can be explained by social and psychological impacts of events;
- cost of restoration of affected industries; and
- cost of imported goods and commodities to substitute the shortage of essential elements.

The calculation, estimation, and aggregation of the above items are altogether known as risk assessment; in contrast, managing the parameters through different management options is considered to be risk management. In considering a set of management options, the loss involving the recovery resilience rate and event size can be formulated as shown in Equation 43:

$$Total\ Loss = f(k_i, n), \quad 43$$

where k_i is the recovery resilience rate among the time. The loss through perturbation in monetary flow among the industries may be written as shown in Equation 44:

$$Total\ Loss_{Monetary\ Flow} = \frac{\sum_{i=1}^M \sum_{t=0}^D (x_i - x_i(t))}{D}, \quad 44$$

where, D is the number of days and, M is the number of industries.

For example, in the numerical example (Figure 12), the assumed mean event size is 6 based on a Poisson distribution. The industry 1's output drops to '0' while industry 2's is not affected directly by the event. There is also no induced initial effect on industry 2. The recovery resilience rate is 0.2. Figure 5.13 demonstrates the random recovery process and also the average loss for each industry over 52 weeks (R was used to run the model).

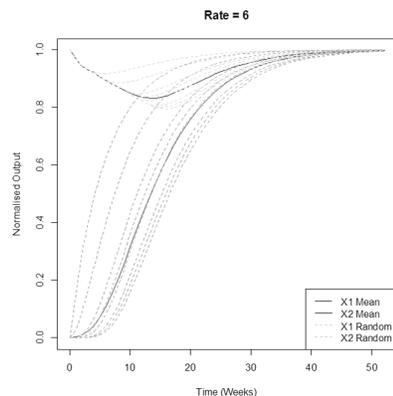


Figure 12: Recovery of the system based random event size

Figure 13 shows the average loss over 52 weeks and the total calculated loss for 52 weeks is 440 million dollars (Industry 1 lost 302.1 million dollars while Industry 2 lost 137.9 million dollars). Figure 13 demonstrates that although the recovery process starts after the event, the total loss increases. This growth in the total loss shows that the event impacts the interconnected industry via the affected industry as well. However, the total loss and recovery find a balance at the pick of the curve and then the total loss decreases to zero.

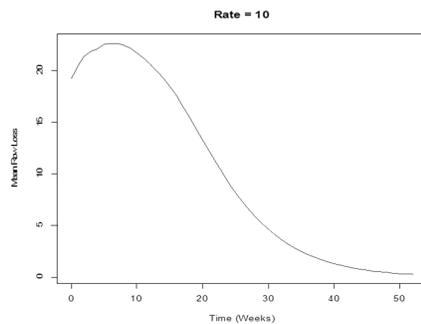


Figure 13: Mean loss in one year

Conclusion

Assessing and subsequently managing the disastrous events whether natural or man-made are essential requirements for managers, planners, and decision makers. Several studies have been done on this issue and numerous models and methods have been proposed and developed in the past. A well-known mathematical model for risk assessment is I/O model exists. The new model developed can be used to explain the interdependency among the industries in an economy. In this paper, the I/O model is improved to make the assessment more accurate and closer to reality. The model is revised through the use of the extreme value and probability theories. The event size is added as a parameter to the model via a mathematically proven process. The newly developed model provides the opportunity to assess and manage the risk by measuring and controlling recovery resilience rate, outage (halting) or recovery time, and event size. This new model can also be used to assess the risk by increasing the size of event using a random process. To explain the distribution of event size a truncated Poisson distribution is used. The zero-event value is excluded (no-event) and the event size is assumed to be natural numbers only. The method of calculating loss using the new model provides a framework for decision makers to achieve a better understanding of the risk and its economic consequences. This new level of modelling and understanding may help the processes such as: protection process, state of readiness, recovery process; or any combination of the above processes. Finally, the parameters in the mathematical model (i.e. recovery resilience rate, outage or halting time, recovery time, and event size) can be calculated and controlled through the framework presented.

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